Taking a New Contour:
A Novel View on Unit Root Test

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Abstract

In this paper we introduce a new view on the distributions of unit root tests. Taking a contour given by the fixed sum of squares instead of the fixed sample size, we show that the distributions of most commonly used unit root tests such as the ones by Dickey-Fuller (1979, 1981) and Phillips (1987) are normal in large samples. The normal asymptotics along the new contour hold under both the null of a unit root and the local-to-unity alternative. Moreover, our results are applicable also for the models with deterministic components, as long as they are removed recursively using only the past information.

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1. Introduction

It is well known that the distributional theories for many of the commonly used unit root tests are nonstandard. For instance, the Dickey-Fuller and Phillips tests both have nonnormal distributions, which are usually represented by the functionals of Brownian motion and Ornstein-Uhlenbeck process under respectively the unit root null and the local-to-unity alternative. The characteristics of their null distributions, which are often referred to as the Dickey-Fuller distributions named after who first tabulate them, have been studied by several authors including Evans and Savin (1981, 1984) and Abadir (1993). In particular, the Dickey-Fuller distributions are known to be asymmetric and skewed to the left. See, e.g., Fuller (1996).

This paper introduces a novel view on these and other related distributions. The sampling distribution of a statistic is usually obtained for a given sample size. Using the conventional sampling distribution of the statistic for the purpose of statistical inference thus implies that we evaluate the likelihood of a realized value of a statistic along the contour given by the fixed sample size. In this paper, we suggest to take a different contour in obtaining the sampling distribution of the statistic, i.e., the contour that is given by the fixed sum of squares. In order to assess the likelihood of the statistic, we therefore look for other possible realizations with their sum of squares, rather than their sample sizes, holding fixed.

For the observations from stationary time series, the sum of squares becomes a constant multiple of the sample size for large samples. The contours of the equi-sample-size and the equi-squared-sum to evaluate the likelihood of a realized sample are thus virtually identical if the size of the sample is large enough. This is not so for the samples from unit root processes. If normalized as necessary, the sum of squares of the samples from unit root processes remain to be random. For the unit root samples, it would thus yield a new meaning different from the conventional one to evaluate the likelihood of a given realization against all other possible realizations with the same sum of squares.

As an illustration, we provide ten simulated sample paths with equal sample size, and another ten with equal sum of squares, respectively, in Figures 1 and 2.[1] For the equi-sample-size paths provided in Figure 1, the one with the largest sum of squares are presented in the top-left corner and the one with the smallest sum of squares in the bottom-right corner. For the sample paths with equi-squared-sum in Figure 2, the one with the smallest sample size to attain the required squared sum is presented in the top-left corner, and the one with the largest sample size in the bottom-right corner. Figures 1 and 2 represent two different contours we may take to obtain the sampling distributions of the statistics involving unit root processes.

Of course, to select a contour in evaluating the likelihood of a given realization would ultimately be a subjective matter. Here the choice is whether to look at other possible realizations along the contour of the samples *either* of fixed sample size (with varying sums of squares as required to have the same sample size) as in Figure 1, *or* of fixed sum of squares (with varying sample sizes as required to have the same

[1]More precisely, ten sample paths at 5%, ..., 95% percentiles were chosen out of ten thousand realizations. The sample size is fixed at 100 for Figure 1, while we set the sum of squares to be 0.23 times 100 squared for Figure 2. The setting yields the most comparable results for the two contours considered here.

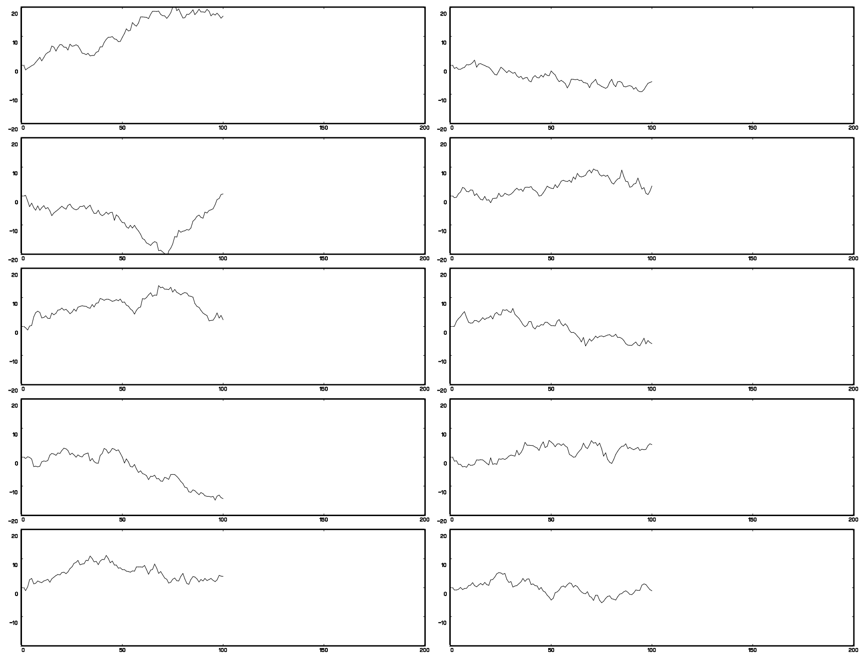

Figure 1: Sample Paths with Equal Sample Size

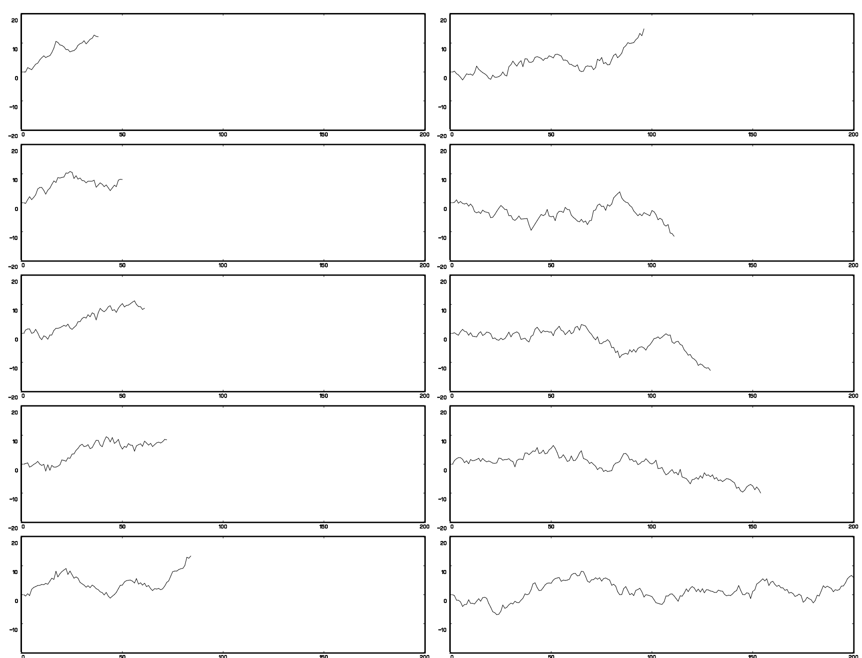

Figure 2: Sample Paths with Equal Sum of Squares

sum of squares) as in Figure 2. Both the size and the sum of squares represent the information contents in the sample on population. Needless to say, the larger dataset and the dataset with larger sampling variations would help us perform more precise inference on the underlying data generating mechanism.

For the unit root tests, we believe that the sum of squares more precisely measures the information content of the samples. The most conspicuous characteristic of the unit root processes (that is constrasted with the stationary process) is the presence of stochastic trend. The degree of conspicuousness of their stochastic trend would essentially lead us to believe, or not to believe, the presence of a unit root in the underlying time series. How conspicuous should they be for us to reject, or not to reject, the unit root hypothesis? Here comes the point that we need a formal statistical test. The remaining question is whether to evaluate the likelihood of a given observation against other possible realizations either of the same sample size exhibiting different degrees of stochastic trends, or having similar degrees of stochastic trends with varying sample sizes. To us, the latter seem more appropriate.

We show that the contour of the equi-squared-sum yields normal asymptotics and conventional statistical theories for the unit root tests. The critical values for the t-ratio can therefore be found from the standard normal table, and all other relevant statistical theories both under the null of a unit root and under the alternative of local-to-unity just follow exactly as in the standard regression model. This is true for models with fitted mean and fitted trends as well, as long as they are removed using only the past information for each observation and the unit root tests are performed in appropriately formulated models. Taking the new contour given by the fixed sum of squares would necessarily make it easier to reject the null hypothesis of a unit root, compared with the conventional approach. The evidence for a unit root in most of macroeconomic time series established in the literature would therefore become weaker, if the new contour is taken.

The rest of the paper is organized as follows. The main results of the paper are given in Section 2. There we consider the prototype unit root model and the test statistic, and develop a new asymptotics along the contour of the equi-squared-sum. The asymptotics are shown to be normal. Section 3 extends our main results into several directions. In particular, it is shown that our main results continue to hold under the local alternatives and for the models with fitted mean and fitted trends. The tests based on more general unit root models are also investigated and shown to yield the normal asymptotics along the new contour. The concluding remarks are given in Section 4, and the mathematical proofs are collected in Appendix. A word on notation. As usual, $\to_d$ and $\to_{a.s.}$ are used to signify respectively the convergence in distribution and the almost sure convergence, and $\sim$ denotes the equivalence in distribution. The standard Brownian motion is denoted by W throughout the paper.

2. Main Results

We consider the autoregressive model

$$y_t = \alpha y_{t-1} + u_t \tag{1}$$

and the test of the unit root hypothesis

$$\alpha = 1 \tag{2}$$

We assume that (u_t) are iid with zero mean and unit (known) second moment. The assumptions are far from being necessary. They are introduced here simply to avoid unnecessary complications and focus on the main issue of the paper. The unknown second moment can be easily estimated consistently from the fitted residuals. Moreover, we may consider more general models, i.e., the models driven by linear processes or weakly dependent innovations, without any difficulty. For such general models, the unit root test may be based on the regression augmented with the lagged differences as for the tests by Dickey and Fuller (1979, 1981), or can be done using the statistic modified nonparametrically as in the tests by Phillips (1987). They all have the same large sample distributions as the test we consider explicitly in the paper, and therefore, our subsequent discussions are also applicable for them. See, e.g., Stock (1994) for the test of a unit root in general models.

Let $y_1, \ldots, y_n$ be the random sample of size n. The unit root hypothesis is routinely tested by the t-ratio on the autoregressive coefficient α, which is given by

$$T_n = \frac{\hat{\alpha}_n - 1}{s(\hat{\alpha}_n)} \tag{3}$$

where $\hat{\alpha}_n = (\sum_{t=1}^n y_{t-1}^2)^{-1} \sum_{t=1}^n y_{t-1} y_t$ is the least squares estimator of α with the standard error $s(\hat{\alpha}_n) = (\sum_{t=1}^n y_{t-1}^2)^{-1/2}$. It is well known that under the null hypothesis of unit root

$$T_n \to_d \left(\int_0^1 W(r)^2 dr \right)^{-1/2} \int_0^1 W(r)\, dW(r) \tag{4}$$

as $n \to \infty$. The limiting distribution in (4), often called the Dickey-Fuller distribution, is nonnormal and skewed to the left. The unit root hypothesis is rejected if T_n takes a large negative value.

We now introduce a new asymptotics. For any given $x > 0$, let $m \geq 1$ be such that

$$m = \inf_{k \geq 1} \left\{ \sum_{t=1}^{k} y_{t-1}^2 \geq x \right\} \tag{5}$$

and consider the t-ratio T_m for the sample of size m. Here the sample size m is determined by the squared sum of (y_t) achieving a certain level. Note that m is a function of (y_t) as well as x.

Theorem 2.1 Assume (2). If we let m_n be defined as in (5) with $x = n^2 c$ for each $n \geq 1$ and some fixed constant $c > 0$, then

$$T_{m_n} \to_d \mathbf{N}(0,1) \tag{6}$$

as $n \to \infty$.

Unlike the conventional result in (4), our approach here yields the normal asymptotics given in (6). A few remarks are now in order.

Remark 2.2 (a) For the choice of

$$c = \frac{1}{n^2} \sum_{t=1}^{n} y_{t-1}^2 \tag{7}$$

we have $m_n = n$ and $T_{m_n} = T_n$. The statistics T_n and T_{m_n} would then have identical values, and may thus be regarded as the same statistic. The large sample distributions in (4) and (6) are derived just by taking two different contours in evaluating the likelihood of a realized value for the statistic. The distribution in (4) is obtained by the conventional approach holding the sample size constant. On the other hand, our new approach yields the distribution in (6) assuming the sum of squares to be constant. The likelihood of a realized value for the statistic is evaluated against other possible realizations from the samples of the same size (with varying sums of squares) and of the same sum of squares (with varying sizes of samples), respectively in (4) and (6).

(b) The samples from stationary time series would produce the same sampling distributions for the two different contours considered above. For the stationary samples, $\sum_{t=1}^{n} y_{t-1}^2/n$ converges to a fixed constant as the sample size grows, due to the law of large numbers, making the two contours identical in large samples. However, the two contours can be very different for the samples from the unit root process. Most of all, the first contour is fixed and nonrandom, whereas the second contour is path-dependent. As is well known,

$$\frac{1}{n^2} \sum_{t=1}^{n} y_{t-1}^2 \to_d \int_0^1 W(r)^2 \, dr$$

for the unit root process. The sum of squares, if normalized properly, would thus remain to be random and depend upon a realized value of the underlying process.

(c) Depending upon which contour we choose to evaluate the likelihood of a realized value for the statistic, the relevant null distribution and thus the critical value of the test would be different. If the realized value of the statistic is to be compared with all of its possible values obtained from the samples of the same size, the critical value from the Dickey-Fuller distribution should be used. If, on the other hand, the realized value of the statistic is to be compared with all possible values given by the samples of the same sum of squares, the standard normal critical value should be used.

(d) The choice of the contour would ultimately be a subjective matter. However, we may say that it would be more appropriate to choose the contour representing the same amount of information on the hypothesis to be tested. In this regard, the contour of the equi-squared-sum is especially appealing for the test of a unit root. The most important distinguishing charactersitic of the sample path from the unit root process (in comparison with that from the stationary process) is the presence of stochastic trend, and its magnitude can be effectively measured by the sum of squares. Choosing the contour of the equi-squared-sum for the unit root test thus implies that we assess the likelihood of a realized test value against other possible realizations having the stochastic trends of the same magnitude. This seems quite reasonable.

(e) Our asymptotics also help to analyze the nonnormality of the Dickey-Fuller distribution. We may clearly see from the proof of Theorem 2.1 that, for a stopping time τ such that $\int_0^\tau W(r)^2 dr$ is constant, the distribution of $\int_0^\tau W(r)\, dW(r)/(\int_0^\tau W(r)^2 dr)^{1/2}$ is standard normal. The nonnormality of the Dickey-Fuller distribution is due to the evaluation of the integrals over the fixed interval $[0, 1]$, rather than the random interval $[0, \tau]$, in the limiting t-ratio.

In Figure 3, the densities for the distributions of T_n and T_{m_n} are given and compared with the standard normal distribution. The densities of T_n are obtained for each of the fixed sample sizes $n = 10, 25, 50$ and 100, while the densities of T_{m_n} are computed for the fixed sum of squares given by $n^2 c$ with $n = 10, 25, 50, 100$ and $c = 0.23$. From simulations, we find that the asymptotic expected value of the stopping time τ defined by $\int_0^\tau W(r)^2 dr = c$ is approximately unity with this choice of c. The densities, in all cases, are quite insensitive to the choice of value of c. Along the contour of the fixed sum of squares, the finite sample distribution of T_{m_n} appears to converge rather rapidly. Our normal asymptotics thus provide very good approximations for the finite sample distributions of T_{m_n}. Even for moderate size samples, the finite sample distributions are indeed quite close to standard normal. In contrast, the distributions of T_n are quite distinct from standard normal at all sample sizes.

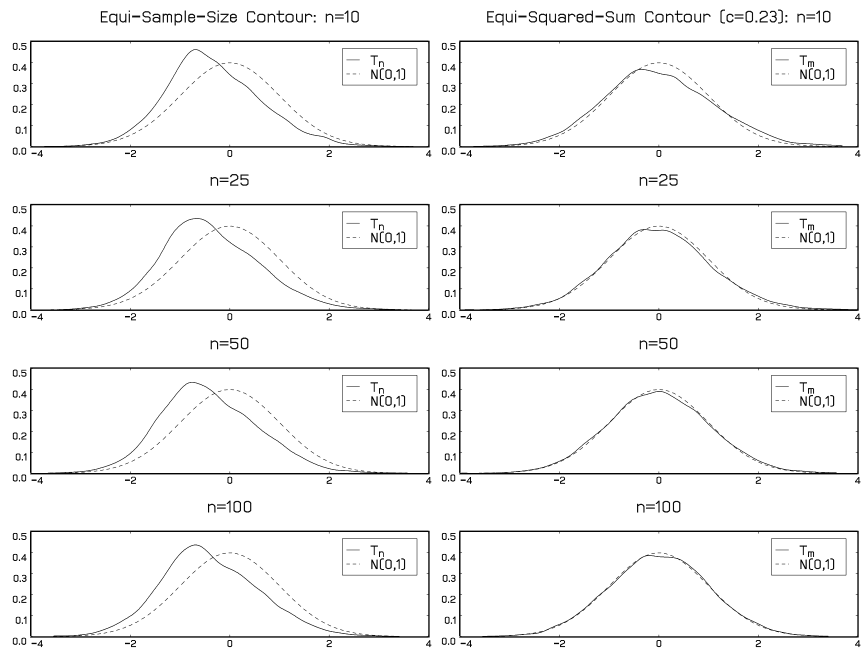

Figure 3: Densities of t-ratios from Equi-Sample-Size and Equi-Squared-Sum Contours

3. Extensions

3.1 Distributions under Local Alternatives

We now consider the local alternative

$$\alpha = 1 - \frac{\delta}{n} \tag{8}$$

for some $\delta > 0$. It is well known that

$$T_n \to_d -\left(\int_0^1 W_\delta(r)^2 dr\right)^{1/2} \delta + \frac{\displaystyle\int_0^1 W_\delta(r)\, dW(r)}{\left(\displaystyle\int_0^1 W_\delta(r)^2 dr\right)^{1/2}} \tag{9}$$

as $n \to \infty$, where W_δ is the Ornstein-Uhlenbeck process given by $W_\delta(r) = \int_0^r \exp[-(r-s)\delta]dW(s)$.

In contrast to the conventional asymptotics in (9), our new asymptotics yield

Corollary 3.1 Assume (8). If we let m_n be defined as in (5) with $x = n^2 c$ for each $n \geq 1$ and some fixed constant $c > 0$, then

$$T_{m_n} \to_d -c^{1/2}\delta + \mathbf{N}(0,1) \tag{10}$$

as $n \to \infty$.

If we take the contour of the fixed sum of squares, we would thus get the standard normal limiting distribution theory under both the null and alternative hypotheses. The unit root t-ratio is distributed as standard normal under the null in large samples. Moreover, it is also normal in large samples under the local alternative, with mean shifted by a constant multiple of the locality parameter. Note that the constant c in (10) is given by (7) for the realized sample of size n. Against the local alternatives, the unit root test is expected to have more powers for the samples with large sums of squares. As is evidently seen from (10) and (7), the large sum of squares has a magnifying effect on the locality parameter.

3.2 Models with Intercept and Time Trend

Our normal asymptotics on the contour of the equi-squared-sum extend well to the models with intercept and linear time trend, if it is removed effectively using only the past information. In this case, the unit root test along the new contour can be based on the regression

$$\triangle^* y_t = (\alpha - 1) y^*_{t-1} + e_t \tag{11}$$

where $(\triangle^* y_t)$ and (y^*_{t-1}) are demeaned or detrended $(\triangle y_t)$ and (y_{t-1}) that are defined more precisely below. More dynamics can be introduced and AR(p), instead of AR(1) in (11), can be used as we explain later.

First, we look at the model with intercept. To test for the unit root in (y_t) generated as

$$y_t = \mu + y_t^\circ$$

where (y_t°) follows the autoregressive process given in (1), we use (y_t^μ) given by

$$y_t^\mu = y_t - y_0 \tag{12}$$

or

$$y_t^\mu = y_t - \frac{1}{t-1} \sum_{k=1}^{t-1} y_k \tag{13}$$

which is defined recursively for each $t = 1, \ldots, n$. This recursive demeaning was first proposed by So and Shin (1999) to demean positively correlated stationary AR processes,[2] and later used in Chang (2002) for the test of the unit root using the nonlinear instrumental variable methodology.

The test for the unit root in (y_t°) can be based on the regression (11) with $y^*_{t-1} = y^\mu_{t-1}$ and $\triangle^* y_t = \triangle y_t$. The conventional limit distribution of the t-ratio T_n^μ for the unit root hypothesis in (11) is dependent upon the actual demeaning procedure that

[2]They found that the recursive demeaning reduces the biases of the parameter estimators.

we introduce in (12) and (13). If (y_t^μ) given in (12) is used, then the limit distribution of T_n^μ is precisely the same as T_n without intercept given in (4). On the other hand, if (y_t^μ) in (13) is used, then the conventional asymptotics would yield

$$T_n^\mu \to_d \left(\int_0^1 W^\mu(r)^2 dr\right)^{-1/2} \int_0^1 W^\mu(r)\, dW(r)$$

where

$$W^\mu(r) = W(r) - \frac{1}{r}\int_0^r W(s)ds$$

as $n \to \infty$.[3]

Now we consider the model with linear time trend, which we write as

$$y_t = \mu + \nu t + y_t^\circ$$

The recursive detrending of (y_t) can be done to obtain

$$y_t^\tau = y_t - y_0 - \sum_{k=1}^t \frac{1}{k}(y_k - y_0) \tag{14}$$

or

$$y_t^\tau = y_t + \frac{2}{t}\sum_{k=1}^t y_k - \frac{6}{t(t+1)}\sum_{k=1}^t k y_k \tag{15}$$

There can be many other alternatives. The regression (11) may now be fitted with $y_{t-1}^* = y_{t-1}^\tau$ and $\triangle^* y_t = \triangle y_t - (y_n - y_0)/n$.

If we denote by T_n^τ the t-ratio for the unit root hypothesis in regression (11), then we have under the conventional asymptotics

$$T_n^\tau \to_d \left(\int_0^1 W^\tau(r)^2 dr\right)^{-1/2} \int_0^1 W^\tau(r)\, dW(r)$$

as $n \to \infty$, where W^τ is given by

$$W^\tau(r) = W(r) - \int_0^r \frac{1}{s} W(s)ds$$

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[3]It follows from the well known Brownian law of iterated logarithm [see, for example, Revuz and Yor (1994, p.53)] that

$$\frac{1}{r}\int_0^r W(s)ds = O(r^{3/2}(\log\log(1/r))^{1/2}) \text{ a.s.}$$

and therefore

$$\frac{1}{r}\int_0^r W(s)ds \to 0 \text{ as } r \to 0$$

The process $W^\mu(r)$ then becomes a continuous stochastic processes defined on $[0,\infty)$, if we let $W^\mu(0) = 0$.

or

$$W^{\tau}(r) = W(r) + \frac{2}{r}\int_0^r W(s)\,ds - \frac{6}{r^2}\int_0^r sW(s)\,ds$$

respectively for (y_t^{τ}) given in (14) or (15).[4]

We now define a new contour

$$m = \inf_{k\geq 1}\left\{\sum_{t=1}^{k} y_{t-1}^{*2} \geq x\right\} \tag{16}$$

similarly as in (5), where $y_{t-1}^{*} = y_{t-1}^{\mu}$ or y_{t-1}^{τ} respectively for the models with intercept and linear time trend. Then we have

Corollary 3.2 Assume (1) and (2). If we let m_n be defined as in (16) with $x = n^2c$ for each $n \geq 1$ and some fixed constant $c > 0$, then

$$T_{m_n}^{\mu}, T_{m_n}^{\tau} \to_d \mathbf{N}(0,1)$$

as $n \to \infty$.

Our previous results therefore also apply for the models with intercept and linear time trend. To obtain the normal asymptotics for the models with fitted mean and fitted linear time trend, however, it is important to use only the past information for demeaning and detrending. The normal asymptotics on the contour of the equi-squared-sum do not follow if the usual demeaning or detrending is used.

3.3 Tests in General Unit Root Models

All our previous results may be easily and naturally extended to more general unit root models. For the test of a unit root in the AR(p) model, we may consider the regression

$$y_t = \alpha y_{t-1} + \sum_{k=1}^{p-1} \alpha_k \triangle y_{t-k} + \varepsilon_t \tag{17}$$

and test whether $\alpha = 1$ using the t-ratio. This is well known. In this case, we let

$$x_t = (\triangle y_{t-1}, \ldots, \triangle y_{t-p+1})'$$

and define

$$y_{p,t} = y_t - \left(\sum_{t=1}^{n} y_t x_t'\right)\left(\sum_{t=1}^{n} x_t x_t'\right)^{-1} x_t$$

[4]Exactly as in the previous footnote, the processes introduced here are well defined to be continuous processes if we set them zero at the origin.

Then the new contour for the t-ratio defined similarly as in (3) for regression (17) is given by

$$m = \inf_{k \geq 1} \left\{ \sum_{t=1}^{k} y_{p,t-1}^2 \geq x \right\} \tag{18}$$

in place of (5). If we denote by $T_{m_n}^p$ the t-ratio based on regression (17) using the sample of size m given in (18), then it can be readily shown that our earlier result continues to apply.

Corollary 3.3 Assume (17) and (2). If we let m_n be defined as in (18) with $x = n^2 c$ for each $n \geq 1$ and some fixed constant $c > 0$, then

$$T_{m_n}^p \to_d \mathbf{N}(0, 1)$$

as $n \to \infty$.

We may show that the statistic $T_{m_n}^p$ has the same distribution as given in Corollary 3.1 of Section 3.1 under the local alternative (8). Moreover, the fitted mean and fitted trends can be allowed and treated exactly as in Section 3.2, and the result in Corollary 3.2 holds also for the statistic $T_{m_n}^p$. More precisely, the unit root test can be based on

$$\triangle^* y_t = (\alpha - 1) y_{t-1}^* + \sum_{k=1}^{p-1} \alpha_k \triangle^* y_{t-k} + e_t$$

similarly as in (11), where (y_t^*) and $(\triangle^* y_t)$ (with y_{t-1}^* and $\triangle^* y_{t-k}$ defined to be the lags of y_t^* and $\triangle^* y_t$, respectively) are given exactly as in Section 3.2 for the models with intercept and linear time trend.

As is well known, the unit root test based on the AR(p) model (17) is valid for more general underlying processes if we let the order p of the AR model increase as the sample size gets large. This was first noted by Said and Dickey (1984), who show that the test based on the standard t-ratio is valid for general invertible ARMA processes of unknown order if we set $p = c\, n^\kappa$ with some constant $c > 0$ and $0 < \kappa \leq 1/3$. More recently, Chang and Park (2003) show that the procedure is indeed valid for more general linear processes with minimum summability condition on their coefficients and under much weaker condition $p = o(n^{1/2})$ on the rate of increase for the fitted AR orders. It can be shown that the result by Chang and Park (2003) continue to hold if we take the new contour. Therefore, the proposed procedure exploiting the new contour is applicable for a broad range of time series models under very mild conditions.

4. Conclusion

In this paper, we develop new asymptotics for the unit root tests that are commonly used in practical applications. Our asymptotics take a new contour given by the fixed sum of squares, and contrast with the conventional ones which evaluate the likelihood of the realized test value along the contour of the fixed sample size. We show in the paper that if the equi-squared-sum contour is chosen, the distribution theories for the tests are normal. They have normal limiting distributions, and we may therefore use the standard normal table for their critical values. As is well known, their conventional asymptotics are nonstandard and nonnormal. Our theories developed in this paper make it clear that we may legitimately use the standard normal table for many of the commonly used unit root tests. It would lead us *not* to making an invalid inference, *but* to exploring a new contour that has never been uncovered.

Appendix: Proofs of Theorems

Proof of Theorem 2.1 Assume (2). Define $W_n(r) = n^{-1/2} y_{[nr]}$, where $[x]$ denotes the largest integer not exceeding $x \geq 0$. It is well known that $W_n \to_d W$ in the space $D(\mathbf{R})$ of cadlag functions endowed with the supremum norm. Moreover, by extending the underlying probability space if necessary, we may assume that W_n and W are defined in the same probability space and that $W_n \to_{a.s} W$ uniformly. Such a construction is possible for instance by the Skorohod embedding. See Hall and Heyde (1980) for details.

For any fixed constant $c > 0$, we let $\tau_n(c)$ be given by

$$\int_0^{\tau_n(c)} W_n(r)^2 dr = c$$

and define a stopping time $\tau(c)$ to be such that

$$\int_0^{\tau(c)} W(r)^2 dr = c$$

Since $W_n \to_{a.s.} W$ uniformly, we have

$$\tau_n(c) \to_{a.s.} \tau(c)$$

as $n \to \infty$. Moreover, upon noticing $\tau_n(c) = m_n/n + O(n^{-1})$ a.s., we may further conclude that

$$\frac{m_n}{n} \to_{a.s.} \tau(c)$$

as $n \to \infty$.

Under the null hypothesis of unit root, we have

$$\begin{aligned} T_{m_n} &= \left(\sum_{t=1}^{m_n} y_{t-1}^2\right)^{-1/2} \sum_{t=1}^{m_n} y_{t-1}u_t \\ &= \left(\int_0^{m_n/n} W_n(r)^2 dr\right)^{-1/2} \int_0^{m_n/n} W_n(r)\, dW_n(r) \\ &= \left(\int_0^{\tau(c)} W(r)^2 dr\right)^{-1/2} \int_0^{\tau(c)} W(r)\, dW(r) + o(1) \text{ a.s.} \end{aligned} \tag{19}$$

as $n \to \infty$, since $W_n \to_{a.s.} W$ uniformly and $m_n/n \to_{a.s.} \tau(c)$ as $n \to \infty$. However, the process V defined by

$$V(s) = \int_0^{\tau(s)} W(r)\, dW(r)$$

is the DDS Brownian motion of the martingale M

$$M(s) = \int_0^{s} W(r)\, dW(r)$$

and therefore,

$$c^{-1/2}V(c) = \left(\int_0^{\tau(c)} W(r)^2 dr\right)^{-1/2} \int_0^{\tau(c)} W(r)\, dW(r) \sim \mathbf{N}(0,1) \tag{20}$$

for any given $c > 0$. The reader is referred to, e.g., Revuz and Yor (1994) for the DDS Brownian motion. The stated result now follows readily from (19) and (20), and the proof is complete. ■

Proof of Corollary 3.1 We use the same notation as in the proof of Theorem 2.1. Assume (8) and let $W_{n\delta}(r) = n^{-1/2}y_{[nr]}$. It follows that $W_{n\delta} \to_d W_\delta$ uniformly in $D(\mathbf{R})$. This is well known. If we define $\tau_\delta(c)$ by

$$\int_0^{\tau_\delta(c)} W_\delta(r)^2 dr = c \tag{21}$$

for a given fixed $c > 0$, then $m_n/n \to_{a.s.} \tau_\delta(c)$ exactly as in the proof of Theorem 2.1.

Under the alternative of local-to-unity, we have

$$
\begin{aligned}
T_{m_n} &= -\left(\sum_{t=1}^{m_n} y_{t-1}^2\right)^{1/2} \frac{\delta}{n} + \left(\sum_{t=1}^{m_n} y_{t-1}^2\right)^{-1/2} \sum_{t=1}^{m_n} y_{t-1}u_t \\
&= -\left(\int_0^{m_n/n} W_{n\delta}(r)^2 dr\right)^{1/2} \delta + \left(\int_0^{m_n/n} W_{n\delta}(r)^2 dr\right)^{-1/2} \int_0^{m_n/n} W_{n\delta}(r)\, dW_n(r) \\
&= -\left(\int_0^{\tau_\delta(c)} W_\delta(r)^2 dr\right)^{1/2} \delta + \left(\int_0^{\tau_\delta(c)} W_\delta(r)^2 dr\right)^{-1/2} \int_0^{\tau_\delta(c)} W_\delta(r)\, dW(r) + o(1) \quad \text{a.s.}
\end{aligned} \tag{22}
$$

as $n \to \infty$. We now consider the DDS Brownian motion

$$V_\delta(s) = \int_0^{\tau_\delta(s)} W_\delta(r)\, dW(r)$$

of the martingale

$$M_\delta(s) = \int_0^{s} W_\delta(r)\, dW(r)$$

from which the stated result follows immediately, due to (21) and (22). ■

Proof of Corollary 3.2 The proof is straightforward given our earlier results, and therefore, omitted. ■

Proof of Corollary 3.3 The proof follows immediately from our previous results. ■

References

Abadir, K.M. (1993). “The limiting distribution of the autocorrelation coefficient under a unit root,” *Annals of Statistics*, 21, 1058-1070.

Chang, Y. (2002). “Nonlinear IV unit root tests in panels with cross-sectional dependency,” *Journal of Econometrics*, 110, 261-292.

Dickey, D. A. and W. A. Fuller (1979). ”Distribution of estimators for autoregressive time series with a unit root,” *Journal of the American Statistical Association* 74, 427-431.

Dickey, D.A. and W.A. Fuller (1981). “Likelihood ratio statistics for autoregressive time series with a unit root,” *Econometrica* 49, 1057-1072.

Evans, G.B.A. and N.E. Savin (1981). "Testing for unit roots: 1," *Econometrica*, 49, 753-779.

Evans, G.B.A. and N.E. Savin (1984). "Testing for unit roots: 2," *Econometrica*, 52, 1241-1269.

Fuller, W.A. (1996). *Introduction to Statistical Time Series*, 2nd ed. Wiley: New York.

Hall, P. and C.C. Heyde (1980). *Martingale Limit Theory and Its Application*. Academic Press: New York.

Phillips, P.C.B. (1987). "Time series regression with a unit root," *Econometrica* 55, 277-301.

So, B.S. and D.W. Shin (1999). "Recursive mean adjustment in time-series inferences," *Statistics and Probability Letters*, 43, 65-73.

Stock, J.H. (1994). "Unit roots and structural breaks," In R.F. Engle and D. McFadden, eds., *Handbook of Econometrics*, Vol. 4, 2739-2841, Elsevier: Amsterdam.

Revuz, D. and M. Yor (1994). *Continuous Martingale and Brownian Motion*, 2nd ed. Springer-Verlag: New York.